

# Adjoint to a Fourier-Mukai functor

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## ABSTRACT

Given a Fourier-Mukai functor  $\Phi$  in the general setting of singular schemes, under various hypotheses we provide both left and a right adjoints to  $\Phi$ , and also give explicit formulas for them. These formulas are simple and natural, and recover the usual formulas when the Fourier-Mukai kernel is a perfect complex. This extends previous work of [1, 15, 16] and has applications to the twist autoequivalences of [10].

## 1. Introduction

Much of the current interest in derived categories, at least from a geometric perspective, began with the seminal result of Orlov [24], who showed in 1997 that any fully faithful functor between the bounded derived categories of two smooth projective varieties  $X$  and  $Y$  is isomorphic to a Fourier-Mukai functor, i.e. of the form

$$\Phi_{\mathcal{P}}^{X \rightarrow Y}(\cdot) = R\pi_{Y*}(L\pi_X^*(\cdot) \overset{L}{\otimes} \mathcal{P}),$$

where  $\pi_X$  and  $\pi_Y$  are projections from  $X \times Y$  to  $X$  and  $Y$  respectively.

More recently, there have been many attempts to improve our understanding of the Bridgeland stability manifold [6, 4] and of the autoequivalence group of a derived category [14, 27]. To construct such autoequivalences, it is convenient to have explicit adjoints; however, for a Fourier-Mukai functor this is an open problem in a general setting.

When  $X$  and  $Y$  are smooth over a field, and the kernel of the Fourier-Mukai functor is a perfect complex, it is well known that the left and right adjoints exist and have a prescribed form. This result has been extended by [1] to the case of separable schemes of finite type over a field. Further, [15] gives a formula for adjoints for projective Gorenstein schemes over a field, under the weaker assumption that the kernel be of “finite homological dimension” over  $X$  and  $Y$ , and [16] gives the right adjoint in the projective case under the same hypothesis for the kernel.

In this paper we prove the existence of both left and right adjoint to a Fourier-Mukai functor, and provide explicit formulas, under considerably weaker hypotheses. Our main result is the following:

**THEOREM 1.1** (See Theorem 3.1 and Theorem 4.1). *Let  $X$  and  $Y$  be separated, quasi-compact schemes, essentially of finite type over a Noetherian scheme  $S$ .*

- (i) *Let either  $Y$  be quasi-projective over  $S$ , or  $Y$  be such that there exists a compactification  $Y \xrightarrow{j} \bar{Y}$  with  $X \times_S \bar{Y}$  of finite Tor-dimension over  $X$ . Let  $\mathcal{P}$  be a  $\pi_X$ -perfect complex in*

$D_{\text{coh}}^b(X \times_S Y)$ . Then  $\Phi_{\mathcal{P}}^{X \rightarrow Y} : D_{\text{Qch}}(X) \rightarrow D_{\text{Qch}}(Y)$  has a left adjoint  $\Phi_{\mathcal{P}_L}^{Y \rightarrow X}$ , where

$$\mathcal{P}_L = \mathcal{H}om_{X \times_S Y}(\mathcal{P}, \pi_X^! \mathcal{O}_X).$$

- (ii) Let either  $X$  be quasi-projective over  $S$ , or  $X$  be such that there exists a compactification  $X \xrightarrow{j} \bar{X}$  with  $\bar{X} \times_S Y$  of finite Tor-dimension over  $Y$ . Let  $\mathcal{P}$  be a  $\pi_Y$ -perfect complex in  $D_{\text{coh}}^b(X \times_S Y)$ . Then  $\Phi_{\mathcal{P}}^{X \rightarrow Y} : D_{\text{Qch}}(X) \rightarrow D_{\text{Qch}}(Y)$  has a right adjoint  $\Phi_{\mathcal{P}_R}^{Y \rightarrow X}$ , where

$$\mathcal{P}_R = \mathcal{H}om_{X \times_S Y}(\mathcal{P}, \pi_Y^! \mathcal{O}_Y).$$

The formulas for the adjoints in the theorem above coincide with the usual formulas when the kernel  $\mathcal{P} \in D_{\text{coh}}^b(X \times_S Y)$  is a perfect complex. In general, however, dualizing is not well-behaved and therefore the formulas stated above are more natural and can be applied in this more general context.

When the given Fourier-Mukai functor is an equivalence of categories, we show that  $\mathcal{P}$  is both  $\pi_X$ - and  $\pi_Y$ -perfect, so that the explicit formulas above do realize the adjoints:

**COROLLARY 1.2.** *Assume that  $X \rightarrow S$  and  $Y \rightarrow S$  are projective. Let  $\mathcal{P} \in D_{\text{Qch}}^b(X \times_S Y)$ . If the functor  $\Phi_{\mathcal{P}}^{X \rightarrow Y}$  gives an equivalence  $D_{\text{coh}}^b(X)$  to  $D_{\text{coh}}^b(Y)$ , then*

- (i)  $\mathcal{P}$  is  $\pi_X$ - and  $\pi_Y$ -perfect
- (ii)  $\Phi_{\mathcal{P}}^{X \rightarrow Y}$  has left and right adjoints given by the formulas of Theorem 1.1.

One of the ingredients in this paper consists of recent results on the twisted inverse image pseudofunctor developed by Lipman [20], Neeman [21] and their co-authors. The most recent of these results [21] hold on the level of the derived category of quasi-coherent sheaves and extend previous results [20] valid on  $D_{\text{Qch}}^+(X)$ . This allowed this paper to be written in the correct generality for the whole  $D_{\text{Qch}}(X)$ .

The results in this paper have applications to the study of flopping contractions and non-commutative deformations of Donovan-Wemyss. Given a projective threefold  $X$  and a connected chain of flopping curves, in the papers [10] and [9] Donovan and Wemyss produce an autoequivalence of the derived category of the threefold associated to the flopping contraction, under some hypotheses on  $X$ . This autoequivalence is related to Bridgeland's flop-flop functor.

At various steps in the construction, the existence of adjoints is required. In the first paper [10] results are developed in the Gorenstein projective setting (see [10, Remark 7.7]). The technology we develop in what follows allows them in the following paper [9] to drop the projectivity assumption as well as relax the conditions on singularities, so that most of their theorems only require Gorenstein terminal singularities in a neighborhood of the contracted curve. .

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## 1.2 Notation and conventions

Throughout this paper,  $X$  and  $Y$  will be separated schemes, essentially of finite type over a Noetherian scheme  $S$ . This implies that  $X$  and  $Y$  are Noetherian and so is  $X \times_S Y$ .

We denote by  $D(X)$  the derived category of  $\mathcal{O}_X$ -modules. The subcategories  $D^+(X)$ , resp.  $D^-(X)$  are the full subcategories of objects  $C \in D(X)$  with  $H^n(C) = 0$  for all  $n \ll 0$ , resp.  $n \gg 0$ ; we set  $D^b(X) = D^+(X) \cap D^-(X)$ . For  $\bullet = \emptyset, +, -, b$  we denote by  $D_{\text{Qch}}^\bullet(X)$ , resp.  $D_{\text{coh}}^\bullet(X)$  the full subcategories whose objects are complexes with quasi-coherent, resp. coherent cohomology.

We will omit the R's and L's in front of derived functors for simplicity. All functors are derived. The derived sheaf Hom is denoted by  $\underline{\mathcal{H}om}$  whereas the set of morphisms in the derived category is indicated by  $\text{Hom}$ .

We denote by  $\pi_X$  and  $\pi_Y$  the projections of the fiber product  $X \times_S Y$  onto its factors. For  $\mathcal{P} \in D_{\text{coh}}^b(X \times_S Y)$ , the Fourier-Mukai functor with kernel  $\mathcal{P}$  is defined as the functor

$$\Phi_{\mathcal{P}}^{X \rightarrow Y}(\cdot) = \pi_{Y*}(\pi_X^*(\cdot) \otimes \mathcal{P})$$

mapping  $D_{\text{Qch}}(X)$  to  $D_{\text{Qch}}(Y)$ .

## 2. Preliminaries

For the reader's convenience, we will state here some well-known fact about functors in derived categories.

LEMMA 2.1. [25, Theorem A] *If  $X$  is a scheme, and  $\mathcal{F}, \mathcal{G}, \mathcal{H} \in D(X)$ , then there is a functorial isomorphism*

$$\underline{\mathcal{H}om}_X(\mathcal{F}, \underline{\mathcal{H}om}_X(\mathcal{G}, \mathcal{H})) \cong \underline{\mathcal{H}om}_X(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}).$$

For a morphism of schemes  $f : X \rightarrow Y$ ,  $\mathcal{F} \in D(Y)$ ,  $\mathcal{G} \in D(X)$ , we have

$$\underline{\mathcal{H}om}_Y(\mathcal{F}, f_*\mathcal{G}) \cong f_* \underline{\mathcal{H}om}_X(f^*\mathcal{F}, \mathcal{G})$$

We collect in the following lemma some basic facts about open immersions:

LEMMA 2.2. *Let  $X$  be quasi-compact, quasi-separated and let  $U \xrightarrow{j} X$  an open immersion. Then*

- (i)  $j_* \underline{\mathcal{H}om}_U(\mathcal{F}, \mathcal{G}) = \underline{\mathcal{H}om}_X(j_*\mathcal{F}, j_*\mathcal{G})$  for all  $\mathcal{F}, \mathcal{G} \in D(U)$
- (ii)  $j_*(\mathcal{F} \otimes \mathcal{G}) = j_*\mathcal{F} \otimes j_*\mathcal{G}$  for all  $\mathcal{F}, \mathcal{G} \in D(U)$
- (iii)  $j^* \underline{\mathcal{H}om}_X(\mathcal{F}, \mathcal{G}) = \underline{\mathcal{H}om}_U(j^*\mathcal{F}, j^*\mathcal{G})$  for all  $\mathcal{F}, \mathcal{G} \in D(X)$

*Proof.* The first two identities are well known and follow from the fact that for an open immersion we have an isomorphism  $j^*j_* \cong \text{id}$ . For example, to see (1) observe that

$$j_* \underline{\mathcal{H}om}_U(\mathcal{F}, \mathcal{G}) = j_* \underline{\mathcal{H}om}_U(j^*j_*\mathcal{F}, \mathcal{G}) = \underline{\mathcal{H}om}_X(j_*\mathcal{F}, j_*\mathcal{G})$$

by Lemma 2.1. The second assertion can be proved in a similar way.

The third identity is [20, Proposition 4.6.7]. □

### 2.1 The twisted inverse image pseudofunctor

We use the notation from [20]. By [21, Theorem 1.2], for a separated morphism  $f : X \rightarrow Y$  essentially of finite type, there exists a well-defined functor  $f^! : D_{\text{Qch}}(Y) \rightarrow D_{\text{Qch}}(X)$ . If  $f : X \rightarrow Y$  is proper then  $f^!$  coincides with  $f^\times$ , the right adjoint to the derived direct image functor  $Rf_*$ . If  $j$  is an open immersion then  $j^! = j^*$ . For the composition of two separated morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z$  essentially of finite type, we have that  $(gf)^! = f^!g^!$  if one of the conditions below holds:

- $f$  is of finite Tor-dimension.
- The composite  $gf$  is proper.
- We restrict to the subcategory  $D_{\text{Qch}}^+(Z) \subset D_{\text{Qch}}(Z)$ .

LEMMA 2.3. [21, Proposition 7.11] *If  $f : X \rightarrow Y$  is a separated morphism essentially of finite type,  $E$  is a perfect complex in  $D_{\text{Qch}}(Y)$  and  $F \in D_{\text{Qch}}(Y)$  is arbitrary, then the map*

$$\sigma(f, E, F) : f^*E \otimes f^!F \rightarrow f^!(E \otimes F)$$

*is an isomorphism. If  $f$  is of finite Tor-dimension then  $\sigma(f, E, F)$  is an isomorphism for all  $E, F \in D_{\text{Qch}}(Y)$ .*

## 2.2 Relative perfect complexes

We remind the reader of the following definition:

DEFINITION 2.4. [18, Definition 4.1] *Let  $f : X \rightarrow Y$  be a morphism. A complex  $F \in D_{\text{coh}}^b(X)$  is said to be perfect relative to  $f$  if it is locally isomorphic in the derived category of  $f^{-1}\mathcal{O}_Y$ -modules to a bounded complex of flat  $f^{-1}\mathcal{O}_Y$ -modules.*

In particular,  $f$  is perfect if  $\mathcal{O}_X$  is perfect over  $f$ . Perfection over  $\text{id}_X$  is equivalent to perfection in  $D(X)$ . Moreover, if  $f$  is a perfect morphism then a perfect complex in  $D_{\text{coh}}^b(X)$  is  $f$ -perfect (see [18, Corollaire 4.5.1]). More generally, if  $f$  is locally of finite type and finite Tor-dimension and  $Y$  is Noetherian, then by [18, Example 1.3] and [18, Corollaire 4.5.1] a perfect complex in  $D_{\text{coh}}^b(X)$  is perfect with respect to  $f$ .

REMARK 2.5. *This notion satisfies two-out-of-three (see [18, Notation 4.2]).*

LEMMA 2.6. [18, Corollaire 3.4] *Let  $f : X \rightarrow Y$  be a morphism of schemes. If  $\mathcal{F}$  is a perfect complex on  $X$  and  $\mathcal{E}$  is perfect with respect to  $f$ , then  $\mathcal{F} \otimes \mathcal{E}$  is perfect with respect to  $f$ .*

Before we continue, let us give some examples of complexes that are perfect relative to a morphism. We will use the following:

PROPOSITION 2.7. [18, Corollaire 3.7.1, Proposition 4.8] *Let  $S$  be a locally Noetherian scheme, and consider a diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \swarrow h \\ & S & \end{array}$$

*Then*

- (i) *If  $f$  is a quasi-compact, quasi-separated morphism of schemes that are quasi-compact over  $S$ , and  $E \in D_{\text{Qch}}^b(X)$  is perfect with respect to  $g$ , then  $f_*E$  is perfect with respect to  $h$ .*
- (ii) *If  $f$  is proper,  $X$  and  $Y$  are locally of finite type over  $S$ , and  $E \in D_{\text{Qch}}^b(X)$  is perfect with respect to  $g$ , then  $f_*E$  is perfect with respect to  $h$ .*

EXAMPLE 2.8. *If  $X$  is locally of finite type over  $S$ , then  $\mathcal{O}_\Delta \in D^b(X \times_S X)$  is  $\pi_X$ -perfect, where  $\pi_X$  is the projection from  $X \times_S X$  to one of the factors. This follows from Proposition 2.7 with  $Y = X \times_S X$ ,  $S = X$ ,  $f : X \rightarrow X \times_S X$  the diagonal embedding,  $g = \text{id}$ ,  $h = \pi_X$ .*

EXAMPLE 2.9. If  $C \xrightarrow{i} X \times_S X$  is a closed immersion, again by Proposition 2.7 we have that  $i_*\mathcal{O}_C$  is  $\pi_X$ -perfect if the composition  $\pi_X \circ i : C \rightarrow X$  is a perfect morphism.

Perfect complexes and  $f$ -perfect complexes behave well under  $\underline{\mathcal{H}om}$ :

PROPOSITION 2.10. [2, Proposition 2.3.9] Let  $f : X \rightarrow Y$  be a morphism of schemes. If  $\mathcal{E}$  is a perfect complex on  $Y$  then  $\underline{\mathcal{H}om}(-, f^!\mathcal{E})$  takes  $f$ -perfect complexes to  $f$ -perfect complexes.

DEFINITION 2.11. [2, Definition 1.3.1] Given  $\mathcal{F} \in D(X)$  and  $\mathcal{P} \in D_{\text{coh}}^b(X)$ , we say that  $\mathcal{P}$  is derived  $\mathcal{F}$ -reflexive if  $\underline{\mathcal{H}om}(\mathcal{P}, \mathcal{F}) \in D_{\text{coh}}^b(X)$  and the canonical morphism

$$\delta_{\mathcal{F}}^{\mathcal{P}} : \mathcal{P} \rightarrow \underline{\mathcal{H}om}_X(\underline{\mathcal{H}om}_X(\mathcal{P}, \mathcal{F}), \mathcal{F})$$

corresponding via Lemma 2.1 to the natural composition

$$\mathcal{P} \otimes_X \underline{\mathcal{H}om}_X(\mathcal{P}, \mathcal{F}) \xrightarrow{\cong} \underline{\mathcal{H}om}_X(\mathcal{P}, \mathcal{F}) \otimes_X \mathcal{P} \xrightarrow{\epsilon} \mathcal{F},$$

where  $\epsilon$  is the evaluation map, is an isomorphism.

LEMMA 2.12. [18, Corollaire 4.9.2] If  $f : X \rightarrow Y$  is a compactifiable morphism, then each complex that is perfect over  $f$  is derived  $f^!\mathcal{O}_Y$ -reflexive.

In particular, by [8] this is true for a separated morphism of finite type between quasi-compact quasi-separated schemes.

LEMMA 2.13. [3, Lemma 6.6] Let  $X, Y$  be Noetherian schemes,  $g : X \rightarrow Y$  be a morphism of finite Tor dimension. Let  $\mathcal{P}$  be a  $g$ -perfect complex. Let  $\mathcal{E}$  a perfect complex over  $Y$  and  $\mathcal{F} \in D_{\text{Qch}}(Y)$ . Then we have an isomorphism

$$\underline{\mathcal{H}om}_X(\mathcal{P}, g^!\mathcal{E}) \otimes_X g^*\mathcal{F} \cong \underline{\mathcal{H}om}_X(\mathcal{P}, g^!\mathcal{E} \otimes g^*\mathcal{F})$$

In the given reference the authors give a proof under the more restrictive, but unnecessary, hypothesis that  $\mathcal{F} \in D_{\text{Qch}}^+(Y)$ . This is solely because the general technology was not available at the time.

### 2.3 The projective case

In the projective case, some of the results we mentioned above are true without the hypothesis of finite Tor dimensionality. In particular, we will prove an analogue of Lemma 2.13 in Lemma 2.15 below. This requires the following lemma:

LEMMA 2.14. [16, Proposition 2.5], [15, Lemma 1.4] Let  $f : X \rightarrow Y$  a projective morphism and  $\mathcal{O}(1)$  be a relatively very ample line bundle on  $X$ , let  $\mathcal{E}, \mathcal{F} \in D_{\text{Qch}}(X)$ . Then

- (i)  $\mathcal{E} \in D_{\text{Qch}}^b(X)$  if and only if  $f_*(\mathcal{E}(n)) \in D_{\text{Qch}}^b(Y)$  for every integer  $n$ .
- (ii) A morphism  $g : \mathcal{E} \rightarrow \mathcal{F}$  is an isomorphism if and only if the induced morphism  $f_*(\mathcal{E}(n)) \rightarrow f_*(\mathcal{F}(n))$  is an isomorphism in  $D_{\text{Qch}}(Y)$  for every integer  $n$ .

LEMMA 2.15. [16, Proposition 2.8] Let  $g : X \rightarrow Y$  be a projective morphism. Let  $\mathcal{P}$  be an object in  $D_{\text{coh}}^b(X)$ , perfect with respect to  $Y$ . Let  $\mathcal{F} \in D_{\text{Qch}}(Y)$ . Then we have an isomorphism

$$\underline{\mathcal{H}om}_X(\mathcal{P}, g^!\mathcal{O}_Y) \otimes_X g^*\mathcal{F} \cong \underline{\mathcal{H}om}_X(\mathcal{P}, g^!\mathcal{F}).$$

*Proof.* In the given reference, the concept of perfection with respect to a morphism is replaced by the concept of “finite homological dimension” (and the authors ask for  $\mathcal{F} \in D_{\text{coh}}^b(Y)$ ). For the convenience of the reader we repeat their proof in our setting.

We have natural morphisms

$$\underline{\mathcal{H}om}_X(\mathcal{P}, g^! \mathcal{O}_Y) \otimes_X g^* \mathcal{F} \rightarrow \underline{\mathcal{H}om}_X(\mathcal{P}, g^! \mathcal{O}_Y \otimes_X g^* \mathcal{F}) \rightarrow \underline{\mathcal{H}om}_X(\mathcal{P}, g^! \mathcal{F}),$$

where the first map is as in [3, (5.5.2)], and the second map is defined as in [21, Definition 7.7]. We have to prove that the composition is an isomorphism. By [16, Lemma 2.5], it is enough to show that the induced morphism

$$g_*(\underline{\mathcal{H}om}_X(\mathcal{P}, g^! \mathcal{O}_Y) \otimes_X g^* \mathcal{F} \otimes \mathcal{O}_X(n)) \rightarrow g_*(\underline{\mathcal{H}om}_X(\mathcal{P}, g^! \mathcal{F}) \otimes \mathcal{O}_X(n))$$

is an isomorphism for all  $n$ . Note that

$$\begin{aligned} \underline{\mathcal{H}om}_X(\mathcal{P}, g^! \mathcal{O}_Y) \otimes_X g^* \mathcal{F} \otimes \mathcal{O}_X(n) &= \underline{\mathcal{H}om}_X(\mathcal{P} \otimes \mathcal{O}_X(-n), g^! \mathcal{O}_Y) \otimes g^* \mathcal{F} \\ \underline{\mathcal{H}om}_X(\mathcal{P}, g^! \mathcal{F}) \otimes \mathcal{O}_X(n) &= \underline{\mathcal{H}om}_X(\mathcal{P} \otimes \mathcal{O}_X(-n), g^! \mathcal{F}) \end{aligned}$$

(see for example [2, Lemma 1.4.6]) because  $\mathcal{O}_X(n)$  is perfect.

By [26, Tag 0A9I],  $g^! \mathcal{O}_Y \in D_{\text{Qch}}^+(X)$ , and hence by [26, Tag 0A6H] we have that  $\underline{\mathcal{H}om}_X(\mathcal{P} \otimes \mathcal{O}_X(-n), g^! \mathcal{O}_Y) \in D_{\text{Qch}}(X)$ . Hence by projection formula and Grothendieck duality ([21, Lemma 3.2]) we have to show that

$$\underline{\mathcal{H}om}_Y(g_* \mathcal{P}(-n), \mathcal{O}_Y) \otimes_X \mathcal{F} \rightarrow \underline{\mathcal{H}om}_X(g_* \mathcal{P}(-n), \mathcal{F})$$

is an isomorphism for all  $n$ . Since by Lemma 2.6  $\mathcal{P}(-n)$  is perfect with respect to  $g$ , hence  $g_* \mathcal{P}(-n)$  is perfect by [18, Proposition 4.8], this follows by [2, Lemma 1.4.6].  $\square$

### 3. The left adjoint

**THEOREM 3.1.** *Let either  $Y$  be a quasi-projective scheme over  $S$ , or  $Y$  be such that there exists a compactification  $Y \xrightarrow{j} \bar{Y}$  with  $X \times_S \bar{Y}$  of finite Tor-dimension over  $X$ . Let  $\mathcal{P}$  be a  $\pi_X$ -perfect complex in  $D_{\text{coh}}^b(X \times_S Y)$ . Then*

$$\Phi_{\mathcal{P}}^{X \rightarrow Y} : D_{\text{Qch}}(X) \rightarrow D_{\text{Qch}}(Y)$$

has a left adjoint

$$\Phi_{\mathcal{P}_L}^{Y \rightarrow X} : D_{\text{Qch}}(Y) \rightarrow D_{\text{Qch}}(X),$$

where

$$\mathcal{P}_L = \underline{\mathcal{H}om}_{X \times_S Y}(\mathcal{P}, \pi_X^! \mathcal{O}_X).$$

**REMARK 3.2.** *The finite Tor-dimension condition is satisfied for example when  $X$  and  $Y$  are defined over a field, i.e.  $S = \text{Spec}(k)$ .*

*Proof.* In the quasi-projective setting, let  $Y \rightarrow S$  factor as  $Y \xrightarrow{j} \bar{Y} \rightarrow S$  with  $\bar{Y}$  projective over  $S$  and  $j$  an open immersion. In the finite Tor dimension setting, let  $j$  and  $\bar{Y}$  as in the statement of the theorem. Hence in either case we have a diagram

$$\begin{array}{ccccc} & & X \times_S \bar{Y} & & \\ & \swarrow \pi_X & \uparrow \text{id} \times j & \searrow \pi_Y & \\ & X & X \times_S Y & Y & \bar{Y} \\ & \swarrow \pi_X & \searrow \pi_Y & \xrightarrow{j} & \\ & X & & & \end{array}$$

where  $j$  is an open immersion and  $\bar{\pi}_X$  is either proper and finite Tor-dimension or a projective morphism (since proper and projective morphisms are stable under base change).

We have functorial isomorphisms

$$\begin{aligned}
& \text{Hom}_X(\Phi_{\mathcal{P}_L}^{Y \rightarrow X}(\mathcal{F}), \mathcal{G}) \\
&= \text{Hom}_X(\pi_{X*}(\mathcal{P}_L \otimes \pi_Y^* \mathcal{F}), \mathcal{G}) \\
&= \text{Hom}_X(\bar{\pi}_{X*}(\text{id} \times j)_*(\mathcal{P}_L \otimes \pi_Y^* \mathcal{F}), \mathcal{G}) \tag{L2} \\
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_*(\mathcal{P}_L \otimes \pi_Y^* \mathcal{F}), \bar{\pi}_X^! \mathcal{G}) \tag{L3} \\
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_*(\underline{\mathcal{H}om}_{X \times_S Y}(\mathcal{P}, \pi_X^! \mathcal{O}_X) \otimes \pi_Y^* \mathcal{F}), \bar{\pi}_X^! \mathcal{G}) \tag{L4} \\
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_*(\underline{\mathcal{H}om}_{X \times_S Y}(\mathcal{P}, (\text{id} \times j)^* \bar{\pi}_X^! \mathcal{O}_X) \otimes \pi_Y^* \mathcal{F}), \bar{\pi}_X^! \mathcal{G}) \tag{L5} \\
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_*(\underline{\mathcal{H}om}_{X \times_S Y}((\text{id} \times j)^*(\text{id} \times j)_* \mathcal{P}, (\text{id} \times j)^* \bar{\pi}_X^! \mathcal{O}_X) \otimes \pi_Y^* \mathcal{F}), \bar{\pi}_X^! \mathcal{G}) \tag{L6} \\
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_*((\text{id} \times j)^* \underline{\mathcal{H}om}_{X \times_S \bar{Y}}((\text{id} \times j)_* \mathcal{P}, \bar{\pi}_X^! \mathcal{O}_X) \otimes \pi_Y^* \mathcal{F}), \bar{\pi}_X^! \mathcal{G}) \tag{L7} \\
&= \text{Hom}_{X \times_S \bar{Y}}(\underline{\mathcal{H}om}_{X \times_S \bar{Y}}((\text{id} \times j)_* \mathcal{P}, \bar{\pi}_X^! \mathcal{O}_X) \otimes (\text{id} \times j)_* \pi_Y^* \mathcal{F}, \bar{\pi}_X^! \mathcal{G}) \tag{L8} \\
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_* \pi_Y^* \mathcal{F}, \underline{\mathcal{H}om}_{X \times_S \bar{Y}}(\underline{\mathcal{H}om}_{X \times_S \bar{Y}}((\text{id} \times j)_* \mathcal{P}, \bar{\pi}_X^! \mathcal{O}_X), \bar{\pi}_X^! \mathcal{G})) \tag{L9}
\end{aligned}$$

where (L3) holds because  $\bar{\pi}_{X*}$  is left adjoint to  $\bar{\pi}_X^!$  since  $\bar{\pi}_X$  is proper. The equality (L4) follows by the definition of  $\mathcal{P}_L$ . The equality (L5) holds because  $\pi_X^! = (\bar{\pi}_X \circ (\text{id} \times j))^! = (\text{id} \times j)^! \bar{\pi}_X^!$  since  $\text{id} \times j$  is finite Tor dimension. Equality L6 holds because  $(\text{id} \times j)^*(\text{id} \times j)_* = \text{id}$  by Lemma 2.2. The equality (L7) holds by Lemma 2.2. Then, (L8) is by projection formula and (L9) follows by Lemma 2.1.

Now note that by Proposition 2.7(1),  $(\text{id} \times j)_* \mathcal{P}$  is  $\bar{\pi}_X$ -perfect (an open immersion of Noetherian schemes is quasi-compact); hence by Proposition 2.10,  $\underline{\mathcal{H}om}_{X \times_S \bar{Y}}((\text{id} \times j)_* \mathcal{P}, \bar{\pi}_X^! \mathcal{O}_X)$  is  $\bar{\pi}_X$ -perfect. Hence we can continue our chain of natural isomorphisms:

$$\begin{aligned}
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_* \pi_Y^* \mathcal{F}, \underline{\mathcal{H}om}_{X \times_S \bar{Y}}(\underline{\mathcal{H}om}_{X \times_S \bar{Y}}((\text{id} \times j)_* \mathcal{P}, \bar{\pi}_X^! \mathcal{O}_X), \bar{\pi}_X^! \mathcal{O}_X) \otimes \pi_X^* \mathcal{G}) \tag{L10} \\
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_* \pi_Y^* \mathcal{F}, (\text{id} \times j)_* \mathcal{P} \otimes \bar{\pi}_X^! \mathcal{G}) \tag{L11} \\
&= \text{Hom}_{X \times_S \bar{Y}}((\text{id} \times j)_* \pi_Y^* \mathcal{F}, (\text{id} \times j)_*(\mathcal{P} \otimes (\text{id} \times j)^* \bar{\pi}_X^* \mathcal{G})) \tag{L12} \\
&= \text{Hom}_{X \times_S Y}(\pi_Y^* \mathcal{F}, \mathcal{P} \otimes (\text{id} \times j)^* \bar{\pi}_X^* \mathcal{G}) \tag{L13} \\
&= \text{Hom}_Y(\mathcal{F}, \pi_{Y*}(\mathcal{P} \otimes \pi_X^* \mathcal{G})) \tag{L14} \\
&= \text{Hom}_Y(\mathcal{F}, \Phi_{\mathcal{P}}^{X \rightarrow Y}(\mathcal{G})). \tag{L15}
\end{aligned}$$

The equality (L10) follows by Proposition 2.15 in the projective case; in the finite Tor-dimension case, it follows because

$$\begin{aligned}
& \underline{\mathcal{H}om}_{X \times_S \bar{Y}}(\underline{\mathcal{H}om}_{X \times_S \bar{Y}}((\text{id} \times j)_* \mathcal{P}, \bar{\pi}_X^! \mathcal{O}_X), \bar{\pi}_X^! \mathcal{G}) \\
&= \underline{\mathcal{H}om}_{X \times_S \bar{Y}}(\underline{\mathcal{H}om}_{X \times_S \bar{Y}}((\text{id} \times j)_* \mathcal{P}, \bar{\pi}_X^! \mathcal{O}_X), \bar{\pi}_X^! \mathcal{O}_X \otimes \pi_X^* \mathcal{G}) \\
&= \underline{\mathcal{H}om}_{X \times_S \bar{Y}}(\underline{\mathcal{H}om}_{X \times_S \bar{Y}}((\text{id} \times j)_* \mathcal{P}, \bar{\pi}_X^! \mathcal{O}_X), \bar{\pi}_X^! \mathcal{O}_X) \otimes \pi_X^* \mathcal{G}
\end{aligned}$$

by Lemma 2.3 and Lemma 2.13.

The equality (L11) is by Proposition 2.12, and (L12) is by projection formula. Further, (L13) follows by Lemma 2.2. We conclude with (L14) by adjunction of  $\pi_Y^*$  and  $\pi_{Y*}$ .  $\square$

#### 4. The right adjoint

**THEOREM 4.1.** *Let either  $X$  be a quasi-projective scheme over  $S$ , or  $X$  be such that there exists a compactification  $X \xrightarrow{j} \bar{X}$  with  $\bar{X} \times_S Y$  of finite Tor-dimension over  $Y$ . Let  $\mathcal{P}$  be a  $\pi_Y$ -perfect complex in  $D_{\text{coh}}^b(X \times_S Y)$ . Then the functor*

$$\Phi_{\mathcal{P}}^{X \rightarrow Y} : D_{\text{Qch}}(X) \rightarrow D_{\text{Qch}}(Y)$$

*has a right adjoint*

$$\Phi_{\mathcal{P}_R}^{Y \rightarrow X} : D_{\text{Qch}}(Y) \rightarrow D_{\text{Qch}}(X),$$

where

$$\mathcal{P}_R = \mathcal{H}om_{X \times_S Y}(\mathcal{P}, \pi_Y^! \mathcal{O}_Y).$$

*Proof.* In the quasi-projective setting, let  $X \rightarrow S$  factor as  $X \xrightarrow{j} \bar{X} \rightarrow S$  with  $\bar{X}$  projective over  $S$  and  $j$  an open immersion. In the finite Tor dimension setting, let  $j$  and  $\bar{X}$  be as in the statement of the theorem. Hence in either case we have a diagram

$$\begin{array}{ccccc} & & \bar{X} \times_S Y & & \\ & \swarrow \bar{\pi}_X & \uparrow j \times \text{id} & \searrow \bar{\pi}_Y & \\ & & X \times_S Y & & \\ & \swarrow \pi_X & & \searrow \pi_Y & \\ \bar{X} & \xleftarrow{j} & X & & Y \end{array}$$

where  $j$  is an open immersion and  $\bar{\pi}_Y$  is either proper and finite Tor-dimension or a projective morphism (since proper and projective morphisms are stable under base change). We have functorial isomorphisms

$$\begin{aligned} \text{Hom}_X(\mathcal{G}, \Phi_{\mathcal{P}_R}^{Y \rightarrow X}(\mathcal{F})) &= \text{Hom}_X(\mathcal{G}, \pi_{X*}(\mathcal{P}_R \otimes \pi_Y^* \mathcal{F})) \\ &= \text{Hom}_{X \times_S Y}(\pi_X^* \mathcal{G}, \mathcal{P}_R \otimes \pi_Y^* \mathcal{F}) & (\text{R2}) \\ &= \text{Hom}_{\bar{X} \times_S Y}((j \times \text{id})_* \pi_X^* \mathcal{G}, (j \times \text{id})_*(\mathcal{P}_R \otimes \pi_Y^* \mathcal{F})) & (\text{R3}) \\ &= \text{Hom}_{\bar{X} \times_S Y}((j \times \text{id})_* \pi_X^* \mathcal{G}, (j \times \text{id})_* \mathcal{P}_R \otimes \bar{\pi}_Y^* \mathcal{F}) & (\text{R4}) \end{aligned}$$

where (R2) is by  $\pi_X^*$ - $\pi_{X*}$  adjunction, (R3) is by Lemma 2.2, and (R4) is by projection formula.

Now note that, by Proposition 2.10,  $\mathcal{P}_R$  is also perfect with respect to  $\pi_Y$ , and hence by Proposition 2.7(1),  $(j \otimes \text{id})_* \mathcal{P}_R$  is  $\bar{\pi}_Y$ -perfect (an open immersion of Noetherian schemes is quasi-compact). Hence we can continue our chain of natural isomorphisms:



$$= \mathrm{Hom}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \pi_X^* \mathcal{G}, \underline{\mathrm{Hom}}_{\bar{X} \times_S Y}(\underline{\mathrm{Hom}}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \mathcal{P}_R, \bar{\pi}_Y^! \mathcal{O}_Y), \bar{\pi}_Y^! \mathcal{O}_Y) \otimes \bar{\pi}_Y^* \mathcal{F}) \quad (\text{R5})$$

$$= \mathrm{Hom}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \pi_X^* \mathcal{G}, \underline{\mathrm{Hom}}_{\bar{X} \times_S Y}(\underline{\mathrm{Hom}}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \mathcal{P}_R, \bar{\pi}_Y^! \mathcal{O}_Y), \bar{\pi}_Y^! \mathcal{F})) \quad (\text{R6})$$

$$= \mathrm{Hom}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \pi_X^* \mathcal{G} \otimes \underline{\mathrm{Hom}}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \mathcal{P}_R, \bar{\pi}_Y^! \mathcal{O}_Y), \bar{\pi}_Y^! \mathcal{F}) \quad (\text{R7})$$

$$= \mathrm{Hom}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* (\pi_X^* \mathcal{G} \otimes (j \times \mathrm{id})^* \underline{\mathrm{Hom}}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \mathcal{P}_R, \bar{\pi}_Y^! \mathcal{O}_Y)), \bar{\pi}_Y^! \mathcal{F}) \quad (\text{R8})$$

$$= \mathrm{Hom}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* (\pi_X^* \mathcal{G} \otimes \underline{\mathrm{Hom}}_{X \times_S Y}(\mathcal{P}_R, \pi_Y^! \mathcal{O}_Y)), \bar{\pi}_Y^! \mathcal{F}) \quad (\text{R9})$$

$$= \mathrm{Hom}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* (\pi_X^* \mathcal{G} \otimes \mathcal{P}), \bar{\pi}_Y^! \mathcal{F}) \quad (\text{R10})$$

$$= \mathrm{Hom}_Y(\bar{\pi}_Y^* (j \times \mathrm{id})_* (\pi_X^* \mathcal{G} \otimes \mathcal{P}), \mathcal{F}) \quad (\text{R11})$$

$$= \mathrm{Hom}_Y(\pi_Y^* (\pi_X^* \mathcal{G} \otimes \mathcal{P}), \mathcal{F}) \quad (\text{R12})$$

$$= \mathrm{Hom}_Y(\Phi_{\mathcal{P}}^{X \rightarrow Y}(\mathcal{G}), \mathcal{F}). \quad (\text{R13})$$

Equality (R5) holds by Proposition 2.12. By Proposition 2.10,  $\underline{\mathrm{Hom}}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \mathcal{P}_R, \bar{\pi}_Y^! \mathcal{O}_Y), \bar{\pi}_Y^! \mathcal{O}_Y$  is  $\bar{\pi}_Y$ -perfect, hence equality (R6) follows by Proposition 2.15 in the projective case; in the finite Tor-dimension case, it follows because

$$\begin{aligned} & \underline{\mathrm{Hom}}_{\bar{X} \times_S Y}(\underline{\mathrm{Hom}}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \mathcal{P}_R, \bar{\pi}_Y^! \mathcal{O}_Y), \bar{\pi}_Y^! \mathcal{O}_Y) \otimes \bar{\pi}_Y^* \mathcal{F} \\ &= \underline{\mathrm{Hom}}_{\bar{X} \times_S Y}(\underline{\mathrm{Hom}}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \mathcal{P}_R, \bar{\pi}_Y^! \mathcal{O}_Y), \bar{\pi}_Y^! \mathcal{O}_Y \otimes \bar{\pi}_Y^* \mathcal{F}) \\ &= \underline{\mathrm{Hom}}_{\bar{X} \times_S Y}(\underline{\mathrm{Hom}}_{\bar{X} \times_S Y}((j \times \mathrm{id})_* \mathcal{P}_R, \bar{\pi}_Y^! \mathcal{O}_Y), \bar{\pi}_Y^! \mathcal{F}) \end{aligned}$$

by Lemma 2.13 and Lemma 2.3. Then, (R7) follows by Lemma 2.1 and (R8) is by projection formula. Equality (R9) holds by Lemma 2.2 and (R10) is by definition of  $\mathcal{P}_R$  and Lemma 2.12. Equality (R11) holds by adjunction of  $\bar{\pi}_Y^*$  and  $\bar{\pi}_Y^!$  since  $\bar{\pi}_Y$  is proper.  $\square$

## 5. The projective case

In this last section we provide two corollaries to our main result in the case where the schemes  $X$  and  $Y$  are projective over the base  $S$ . In this case, from the behavior of the Fourier-Mukai functor  $\Phi_{\mathcal{P}}^{X \rightarrow Y}$  we can deduce properties of the kernel  $\mathcal{P}$ :

- (i)  $\Phi_{\mathcal{P}}^{X \rightarrow Y}$  maps  $D_{\mathrm{coh}}^b(X)$  to  $D_{\mathrm{coh}}^b(Y)$  if and only if  $\mathcal{P}$  is  $\pi_X$ -perfect (Corollary 5.3).
- (ii) If  $\Phi_{\mathcal{P}}^{X \rightarrow Y}$  is an equivalence, then  $\mathcal{P}$  is  $\pi_X$ - and  $\pi_Y$ -perfect (Corollary 5.4).

Then in both cases we get a left adjoint as in Theorem 3.1; in the second case, we get an adjoint as in Theorem 4.1.

LEMMA 5.1. *Let  $\mathcal{P} \in D_{\mathrm{Qch}}^b(X \times_S Y)$ . The following are equivalent:*

- (i)  $\mathcal{P} \in D_{\mathrm{coh}}^b(X \times_S Y)$  and it is  $\pi_X$ -perfect.
- (ii) For all  $\mathcal{F} \in D_{\mathrm{coh}}^b(X)$  we have  $\mathcal{P} \otimes \pi_X^* \mathcal{F} \in D_{\mathrm{coh}}^b(X \times_S Y)$ .

*Proof.* Assume that for all  $\mathcal{F} \in D_{\mathrm{coh}}^b(X)$  we have  $\mathcal{P} \otimes \pi_X^* \mathcal{F} \in D_{\mathrm{coh}}^b(X \times_S Y)$ . Then first of all  $\mathcal{P} = \mathcal{P} \otimes \pi_X^* \mathcal{O}_X \in D_{\mathrm{coh}}^b(X \times_S Y)$ .

By [18, Proposition 4.3], it is enough to show that for every point  $(x, y) \in X \times_S Y$  there exist numbers  $a, b$  such that  $\mathcal{P} \in D_{\mathrm{coh}}^{[a, b]}(X \times_S Y)$  in a neighborhood of  $(x, y)$  and

$$\mathrm{tor\,amp}_{\mathcal{O}_{X, x}}(\mathcal{P}_{(x, y)}) \subset [a, b],$$

where  $x = \pi_X((x, y)) \in X$ .

Since  $\mathcal{P}_{(x,y)} \otimes_{\mathcal{O}_{X,x}} M_x$  is bounded for all  $M_x$  (because we know that we land in  $D_{\text{coh}}^b(X \times_S Y)$ ), there exists some  $N$  such that

$$\text{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{P}_{(x,y)}, k(x)) = 0$$

for all  $i \geq N$ , where  $k(x)$  is the residue field at  $x$ .

Now replace  $\mathcal{P}_{(x,y)}$  by a bounded above complex of free  $\mathcal{O}_{X,x}$ -modules

$$\mathcal{Q}_{(x,y)} := \dots \rightarrow \mathcal{Q}_{(x,y)}^n \rightarrow \mathcal{Q}_{(x,y)}^{n+1} \rightarrow \dots \rightarrow \mathcal{Q}_{(x,y)}^b \rightarrow 0.$$

The complex is exact in low enough degree since  $\mathcal{P} \in D_{\text{coh}}^b(X \times_S Y)$ . Let  $a < -N$  be low enough so that  $\mathcal{Q}_{(x,y)}$  is exact in degree  $\leq a+1$ , and truncate the complex by taking  $\mathcal{K}_{(x,y)} = \ker(\mathcal{Q}_{(x,y)}^{a+1} \rightarrow \mathcal{Q}_{(x,y)}^{a+2})$  so that we have a complex

$$0 \rightarrow \mathcal{K}_{(x,y)} \rightarrow \mathcal{Q}_{(x,y)}^{a+1} \rightarrow \mathcal{Q}_{(x,y)}^{a+2} \rightarrow \dots \rightarrow \mathcal{Q}_{(x,y)}^b \rightarrow 0$$

which is also quasi-isomorphic to  $\mathcal{P}_{(x,y)}$ .

Now consider the distinguished triangle  $\mathcal{K}_{(x,y)}[-a-1] \rightarrow \mathcal{Q}_{(x,y)}^{\geq a+1} \rightarrow \mathcal{P}$ . By the long exact sequence on Tor we have

$$\text{Tor}_1^{\mathcal{O}_{X,x}}(\mathcal{K}_{(x,y)}, k(x)) = \text{Tor}_{-a}^{\mathcal{O}_{X,x}}(\mathcal{K}_{(x,y)}[-a-1], k(x)) = \text{Tor}_{-a}^{\mathcal{O}_{X,x}}(\mathcal{Q}_{(x,y)}^{\geq a+1}, k(x)) = 0$$

since  $\mathcal{Q}_{(x,y)}^{\geq a+1}$  is zero in degree  $a$ .

Hence we obtain that  $\mathcal{K}_{(x,y)}$  is also locally free as an  $\mathcal{O}_{X,x}$ -module, hence  $\mathcal{P}_{(x,y)}$  is quasi-isomorphic to a bounded complex of locally free  $\mathcal{O}_{X,x}$ -modules and  $\text{tor amp}_{\mathcal{O}_{X,x}}(\mathcal{P}_{(x,y)}) \subset [a, b]$ .

The other implication is clear.  $\square$

REMARK 5.2. *The notion in Lemma 5.1(2) corresponds to the notion of “finite homological dimension” in [15, 16]. The notion of perfection with respect to a morphism might seem stronger at first sight, but as we just saw, the two notions are actually equivalent.*

COROLLARY 5.3. *Let  $X$  be projective over  $S$ . Let  $\mathcal{P} \in D_{\text{Qch}}^b(X \times_S Y)$ . The functor  $\Phi_{\mathcal{P}}^{X \rightarrow Y}$  maps  $D_{\text{coh}}^b(X)$  to  $D_{\text{coh}}^b(Y)$  if and only if  $\mathcal{P}$  is in  $D_{\text{coh}}^b(X \times_S Y)$  and is  $\pi_X$ -perfect.*

*Proof.* Assume that  $\Phi_{\mathcal{P}}^{X \rightarrow Y}$  maps  $D_{\text{coh}}^b(X)$  to  $D_{\text{coh}}^b(Y)$ . By Lemma 2.14 it follows that the functor  $\pi_X^*(-) \otimes \mathcal{P}$  sends  $D_{\text{coh}}^b(X)$  to  $D_{\text{coh}}^b(X \times_S Y)$ . We conclude by Lemma 5.1. The other implication is clear.  $\square$

COROLLARY 5.4. *Assume that  $X \rightarrow S$  and  $Y \rightarrow S$  are projective. Let  $\mathcal{P} \in D_{\text{Qch}}^b(X \times_S Y)$ . If the functor  $\Phi_{\mathcal{P}}^{X \rightarrow Y}$  gives an equivalence  $D_{\text{coh}}^b(X)$  to  $D_{\text{coh}}^b(Y)$ , then it follows that  $\mathcal{P}$  is  $\pi_X$ - and  $\pi_Y$ -perfect. Hence in particular it has left and right adjoints given by the formulas of Theorems 3.1 and 4.1.*

*Proof.*  $\mathcal{P}$  is in  $D_{\text{coh}}^b(X \times_S Y)$  by Lemma 5.3. By [16, Proposition 2.10] and Lemma 5.1, it follows that  $\mathcal{P}$  is  $\pi_X$ - and  $\pi_Y$ -perfect. We conclude by Theorems 3.1 and 4.1.  $\square$

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